Pivots, Determinants, and Perfect Matchings of Graphs

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Abstract. We give a characterization of the effect of sequences of pivot operations on a graph by relating it to determinants of adjacency matrices. This allows us to deduce that two sequences of pivot operations are equivalent iff they contain the same set S of vertices (modulo two). Moreover, given a set of vertices S, we characterize whether or not such a sequence using precisely the vertices of S exists. We also relate pivots to perfect matchings to obtain a graph-theoretical characterization. Finally, we consider graphs with self-loops to carry over the results to sequences containing both pivots and local complementation operations.

1 Introduction

The operation of local complementation in an undirected graph takes the neighbourhood of a vertex in the graph and replaces that neighbourhood by its graph complement. The related operation of edge local complementation, here called pivoting, can be defined in terms of local complementation. It starts with an edge in the graph and toggles edges based on the way its endpoints are connected to the endpoints of the pivot-edge.

The operations are connected in a natural way to overlap graphs (also called circle graphs [7]). Given a finite set of chords of a circle, the overlap graph contains a vertex for each chord, and two vertices are connected if the corresponding chords cross. Taking out a piece of the perimeter of the circle delimited by the two endpoints of a chord, and reinserting it in reverse, changes the way the cords intersect, and hence changes the associated overlap graph. The effect of this reversal on the overlap graph can be obtained by a local complementation on the vertex corresponding to the chord. Similarly, interchanging two pieces of the perimeter of the circle, each starting at the different endpoints of one common chord and ending at the endpoints of another, can be modelled by a pivot in the overlap graph.

Overlap graphs naturally occur in theories of genetic rearrangements [11, 6], but local complementation and edge local complementation operations are applied in many settings, like the relationships between Eulerian tours, equivalence of certain codes[5], rank-width of graphs[15], and quantum graph states[14].

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In the present paper we are interested in sequences of pivots in arbitrary simple graphs. In defining a single pivot one usually distinguishes three disjoint neighbourhoods in the graph, and edges are updated according to the neighbourhoods to which the endpoints belong. Describing the effect of a sequence of pivot operations in terms of neighbourhood connections is involved – the number of neighbourhoods to consider grows exponentially in the size of the sequence.

It turns out that by considering determinants of adjacency matrices (in the spirit of [8]) we can effectively describe the effect of sequences of pivot operations. Subsequently, we relate this to perfect matchings, a perfect matching is a set of edges that forms a partition of the set of vertices, to obtain a graph-theoretical characterization. A direct proof of the characterization in terms of perfect matchings is given in the appendix. We obtain the surprising result that the connection between two vertices after a series of pivots directly depends on the number (modulo two) of perfect matchings in the subgraph induced by the two vertices and the vertices of the pivot-edges (with 'multiplicity' if vertices occur more than once).

As an immediate consequence we obtain that the result of a sequence of pivots, provided all pivot operations are defined, i.e., based on an edge in the graph to which they are applied, does not depend on the order of the pivots, but only on the nodes involved (plus their cardinality modulo 2). Also, we show that for any applicable sequence of pivot there exists an equivalent *reduced* sequence where each node appears at most once in the sequence. Finally, we consider the case where graphs can have self-loops, and generalize the results for sequences of pivots to sequences having both local complementation operations and pivots.

2   Preliminaries

Usually we write xy for the pair $\{x, y\}$.

We use $\oplus$ to denote both the logical exclusive-or as well as the related operation of symmetric set difference. The operation is $\oplus$ associative: the exclusive or over a sequence of Booleans is true iff an odd number of the arguments is true.

Let A be a $V \times V$ matrix. For a set $X \subseteq V$ we use $A\langle X \rangle$ to denote the submatrix induced by X, which keeps the rows and columns indexed by X.

The determinant of A is defined as $\det(A) = \sum_{\sigma \in \Pi(V)} \mathrm{sgn}(\sigma) \prod_{u \in V} a_{u, \sigma(u)}$, where $\Pi(V)$ is the set of permutations of V, and $\mathrm{sgn}(\sigma)$ is the sign (or parity) of the permutation, which is well defined after choosing an ordering on V. We will mainly consider the determinant over $GF(2)$, i.e., modulo 2, where the signs do not matter. The determinant of the empty matrix is considered to be 1 (contributed by the empty permutation).

Graphs. The graphs we consider here are simple (undirected and without loops and parallel edges). For graph $G = (V, E)$ we use $V(G)$ and $E(G)$ to denote its set of vertices V and set of edges E, respectively.

We define $x \sim_G y$ if either $xy \in E$ or $x = y$. For $X \subseteq V$, we denote the subgraph of G induced by X as $G\langle X \rangle$. Let $N_G(v) = \{w \in V \mid vw \in E\}$ denote the neighbourhood of vertex v in graph G.
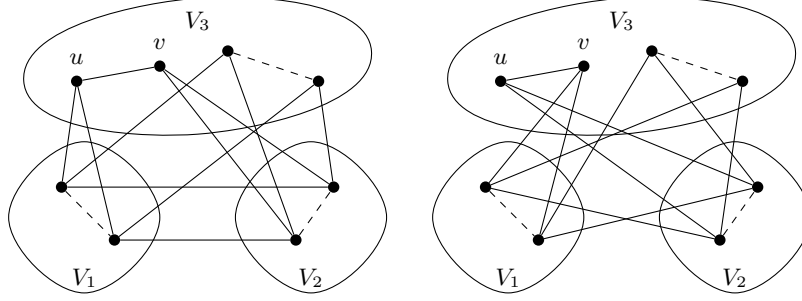
Fig. 1. Pivoting uv. Connection xy is toggled if $x \in V_i$ and $y \in V_j$ with $i \neq j$. Note u and v are connected to all vertices in V_2, these edges are omitted in the diagram. The operation does not effect edges adjacent to vertices outside the sets V_1, V_2, V_3.

With a graph G one associates its adjacency matrix $A(G)$, which is a $V \times V$ $(0,1)$-matrix $(a_{u,v})$ with $a_{u,v} = 1$ iff $uv \in E$. Obviously, for $X \subseteq V$, $A(G\langle X \rangle) = A(G)\langle X \rangle$.

By the determinant of graph G, denoted $\det G$, we will mean the determinant $\det A(G)$ of its adjacency matrix, computed over $GF(2)$.

3 Pivot Operation

Let $G = (V, E)$ be a graph. The graph obtained by *local complementation* at $u \in V$ on G, denoted by $G*u$, is the graph that is obtained from G by complementing the edges in the neighbourhood $N_G(u)$ of u in G. Using a logical expression we can write for $G*u$ the definition $xy \in E(G*u)$ iff $(xy \in E) \oplus (xu \in E \wedge yu \in E)$.

For a vertex x consider its closed neighbourhood $N'_G(x) = N_G(x) \cup \{x\} = \{y \in V_G \mid x \sim_G y\}$. The edge uv partitions the vertices of G connected to u or v into three sets $V_1 = N'_G(u) \setminus N'_G(v)$, $V_2 = N'_G(v) \setminus N'_G(u)$, $V_3 = N'_G(u) \cap N'_G(v)$. Note that $u, v \in V_3$.

Let $uv \in E(G)$. The graph obtained from G by *pivoting* uv, denoted by $G[uv]$, is constructed by 'toggling' all edges between different V_i and V_j: for xy with $x \in V_i$ and $y \in V_j$ $(i \neq j)$: $xy \in E(G)$ iff $xy \notin E(G[uv])$, see Figure 1. The remaining edges remain unchanged. [3]

It turns out that $G[uv]$ equals $G*u*v*u = G*v*u*v$.

Example 1. We start with six segments, of which the relative positions of endpoints can be represented by the string 3 5 2 6 5 4 1 3 6 1 2 4.

The 'entanglement' of these intervals can be represented by the overlap graph to the left in Figure 2. When we pivot on the edge 23 we obtain the graph to the right.

This new graph is the overlap graph of <u>3 6 1 2</u> 6 5 4 1 <u>3 5 2</u> 4.

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[3] In defining this operation usually the description adds the rule that the vertices u and v are swapped. Here this is avoided by including u and v in the set V_3.
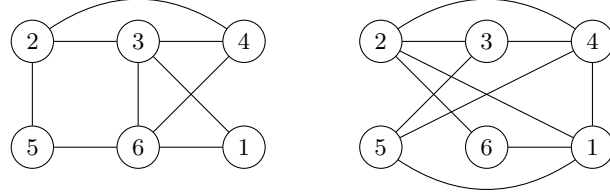
Fig. 2. A graph G and its pivot $G[23]$, cf. Example 1.

$\square$

In order to derive properties of pivoting in an algebraic way, rather than using combinatorial methods in graphs, Oum [15] shows that $G[uv]$ can be described using a logical formula. It turns out that the expression can be stated elegantly in terms of $\sim_G$ rather than in terms of $E(G)$.

Lemma 2. *Let G be a graph, and let $uv \in E(G)$. Then $G[uv]$ is defined by the expression*

$$x \sim_{G[uv]} y = x \sim_G y \oplus ((x \sim_G u) \wedge (y \sim_G v)) \oplus ((x \sim_G v) \wedge (y \sim_G u)).$$

for all $x, y \in V(G)$. $\square$

Pivots and matrices. In a 1997 paper [8] on unimodular $(0,1)$-matrices, Geelen defines a general pivot operation on matrices that is defined for subsets of the indices (thus not only for edges) which turns out to extend the classic pivot operation introduced above.

Let A be a V by V $(0,1)$-matrix, and let $X \subseteq V$ be such that $\det A\langle X \rangle \neq 0$, then the *pivot* of A by X, denoted by $A * X$[4], is defined as follows. If $P = A\langle X \rangle$ and $A = \left(\begin{array}{c|c} P & Q \\ \hline R & S \end{array} \right)$, then

$$A * X = \left(\begin{array}{c|c} -P^{-1} & P^{-1}Q \\ \hline RP^{-1} & S - RP^{-1}Q \end{array} \right).$$

Based on a similar operation from [16] (see also [4, p.230]), the following basic result can be obtained, see [8, Theorem 2.1] and [9] for a full proof.

Proposition 3. *Let A be a $V \times V$ matrix, and let $X \subseteq V$ be such that $\det A\langle X \rangle \neq 0$. Then, for $Y \subseteq V$,*

$$\det(A * X)\langle Y \rangle = \pm \det A\langle X \oplus Y \rangle / \det A\langle X \rangle$$

$\square$

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[4] The local complementation operation G*u differs from $A * \{u\}$ where A is the adjacency matrix, see Section 7

We will apply this result to our (edge) pivots in graphs. Let A be the adjacency matrix of graph G. we start by observing that for vertices $u \neq v$, uv is an edge in G iff the submatrix $A\langle uv \rangle$ is of the form $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ or equivalently $\det G\langle uv \rangle = 1$.

If uv is an edge in G, then (after rearranging rows and columns) A can be written in the form

$$A = \left(\begin{array}{cc|c} 0 & 1 & \chi_u^T \\ \hline 1 & 0 & \chi_v^T \\ \hline \chi_u & \chi_v & A\langle V - u - v \rangle \end{array} \right)$$

where χ_u is the column vector belonging to u without elements a_{uu} and a_{vu}, and, for vector x, x^T is the transpose of x.

As $\det A\langle uv \rangle \neq 0$, the operation pivot $A * uv$ of [8] is well defined. It equals the following matrix which in fact is the matrix of $G[uv]$: the component $(\chi_v \chi_u^T + \chi_u \chi_v^T)$ in the matrix has the same functionality as the expression $((x \sim_G u) \wedge (y \sim_G v)) \oplus ((x \sim_G v) \wedge (y \sim_G u))$ from the characterization of Oum, Lemma 2.

$$A * uv = \left(\begin{array}{cc|c} 0 & 1 & \chi_v^T \\ \hline 1 & 0 & \chi_u^T \\ \hline \chi_v & \chi_u & A\langle V - u - v \rangle - (\chi_v \chi_u^T + \chi_u \chi_v^T) \end{array} \right)$$

We now rephrase the result cited from [8], Proposition 3 above, for pivots in graphs (and where the computations are over $GF(2)$). It will be the main tool in our paper.

Theorem 4. *Let G be a graph, and let $uv \in E(G)$. Then, for $Y \subseteq V(G)$,*

$$\det((G[uv])\langle Y \rangle) = \det(G\langle Y \oplus \{u, v\} \rangle)$$

$\square$

It is noted in Little [13] that over $GF(2)$ the $\det(G) = 0$ has a graph interpretation: $\det(G) = 0$ iff there exists a non-empty set $S \subseteq V(G)$ such that every $v \in V(G)$ is adjacent to an even number of vertices in S. Indeed, S represents a linear dependent set of rows modulo 2.

Finally note that $x \sim_G y$ iff $\det(G[\{x\} \oplus \{y\}]) = 1$. Indeed, if $x = y$, then $\det(G[\emptyset]) = 1$, and if $x \neq y$, then xy is an edge iff $\det(G[xy]) = 1$.

4   Sequences of Pivots

In this section we study series of pivots that are applied consecutively to a graph. It is shown that using determinants there is an elegant formula that describes whether a certain pair of vertices is adjacent in the final resulting graph. From this result we then conclude that the effect of a sequence of pivots only depends on the vertices involved, and not on the order of the operations. Without determinants, using combinatorical argumentations on graphs, this result seems hard to obtain.

A sequence of pivoting operations $\varphi = [v_1 v_2][v_3 v_4] \cdots [v_{n-1} v_n]$ is *applicable* if each pair $[v_i v_{i+1}]$ in the sequence corresponds to an edge $v_i v_{i+1}$ in the graph obtained at the time of application. For such a sequence we define $\sup(\varphi) = \bigoplus_i \{v_i\}$, the set of vertices that occur an odd number of times in the sequence of operations. This is called the *support* of φ. Note that the support always contains an even number of vertices.

Using the correspondence between pivot operations and determinants of submatrices, we can formulate a condition that specifies the edges present in a graph after a sequence of pivots.

Theorem 5. *Let φ be an applicable sequence of pivoting operations for G, and let $S = \sup(\varphi)$. Then $\det(G\varphi\langle xy \rangle) = \det(G\langle S \oplus \{x, y\}\rangle)$ for $x, y \in V(G)$, $x \neq y$. Consequently, $xy \in E(G\varphi)$ iff this value equals 1.*

Proof. We prove the equality in the statement by induction on the number of pivot operations. When φ is the empty sequence, we read the identity $\det G\langle xy \rangle = \det G\langle \varnothing \oplus \{x, y\}\rangle$.

So assume $\varphi = [uv]\varphi'$. Let $S = \sup(\varphi)$, then $S' = S \oplus \{u, v\}$ is the support of φ'. We apply the induction hypothesis to the applicable sequence φ' in the graph $G[uv]$. Then $\det G\varphi\langle xy \rangle = \det G[uv]\varphi'\langle xy \rangle = \det G[uv]\langle S' \oplus \{x, y\}\rangle$. Now we can apply Theorem 4, to obtain $\det G\langle S' \oplus \{x, y\} \oplus \{u, v\}\rangle$ which obviously equals $\det G\langle S \oplus \{x, y\}\rangle$. □

We now have the following surprising direct consequence of the previous theorem.

Theorem 6. *If φ and φ' are applicable sequences of pivoting operations for G, then $\sup(\varphi) = \sup(\varphi')$ implies $G\varphi = G\varphi'$.*

As a consequence, when calculating the orbit of graphs under the pivot operation, as done in [5], we need not consider every sequence – only those that have different support.

The next lemma shows, as a direct corollary to Theorem 5, that the vertices of the support of an applicable sequence φ induce a subgraph that has a nonzero determinant.

Lemma 7. *Let φ be a sequence of pivots applicable in graph G, and let $S = \sup(\varphi)$. Then $\det G\langle S \rangle = 1$.*

Proof. If S is empty, then indeed $\det G\langle \varnothing \rangle = 1$. Now let S (and φ) be non-empty. Let $\varphi = \varphi'[uv]$, so $S = \sup \varphi' \oplus \{u, v\}$. As φ is applicable, uv must be an edge in $G\varphi'$. By Theorem 5, $\det G\varphi'\langle uv \rangle = \det G\langle S \rangle = 1$. □

Two special cases of Theorem 6 are known from the literature, the triangle equality (involving three vertices) and commutativity (involving four vertices).

The *triangle equality* is a classic result in the theory of pivots. Arratia et al. give a proof [2, Lemma 10] involving certain graphs with 11 vertices. Independently Genest obtains this result in his Thesis [10, Proposition 1.3.5]. The cited work of Oum [15, Proposition 2.5] contains a proof which applies Lemma 2.

Corollary 8. *If u, v, w are three distinct vertices in graph G such that uv and uw are edges. Then $G[uv][vw] = G[uw]$.*

Proof. Note that vw is an edge in $G[uv]$ iff $\det G\langle\{v,w\}\oplus\{u,v\}\rangle = \det G\langle\{w,u\}\rangle = 1$. The latter holds iff uw is an edge in G. Hence the pivots at both sides are applicable, and the result follows from Theorem 6. $\qquad\square$

Another result that fits in our framework is the commutativity of pivots on disjoint sets of nodes. It was obtained by Harju et al. [12] (see also [3]) studying graph operations modelled after gene rearrangements in organisms called ciliates. The property states that two disjoint pivots $[uv]$ and $[wz]$, when applicable in either order, have a result independent of the order in which they are applied.

The next lemma is also proved in Corollary 7 of [14] using linear fractional transformations. Essentially, it states that 'twins' stay 'twins' after pivoting. Here we obtain it as a consequence of Theorem 5.

Lemma 9. *Let v, v' be vertices in graph G such that $v \sim_G x$ iff $v' \sim_G x$ for each vertex x. Then for each applicable sequence φ of pivots, $v \sim_{G\varphi} x$ iff $v' \sim_{G\varphi} x$ for each vertex x.*

Proof. Let $S = \sup(\varphi)$. We have $v \sim_{G\varphi} x$ iff $\det(G\varphi[\{v\}\oplus\{x\}]) = 1$ iff $\det(G[S\oplus\{v\}\oplus\{x\}]) = 1$ iff $\det(G[S\oplus\{v\}\oplus\{x\}\oplus\{v\}\oplus\{v'\}]) = 1$ (since $v \sim_G x$ iff $v' \sim_G x$ for each vertex x) iff $\det(G[S\oplus\{v'\}\oplus\{x\}]) = 1$ iff $\det(G\varphi[\{v'\}\oplus\{x\}]) = 1$ iff $v' \sim_{G\varphi} x$. $\qquad\square$

5 Pivots and Perfect Matchings

There is a direct correspondence between (the parity of) the determinant of a graph and (the parity of) the number of perfect matchings in that graph. This correspondence is explained in a paper by Little [13], which we essentially follow below. We include it in our presentation because it allows us to reformulate some results in terms of a property of the graph itself, rather than a property of the associated adjacency matrix. We also give an application, illustrating that the link to perfect matchings adds some intuition to results from the literature.

We say that a partition P of V is a *pairing of V* if it consists of sets of cardinality two. Let $\mathrm{pair}(V)$ be the set of pairings of V. A *perfect matching* in G is a pairing P of $V(G)$ such that $P \subseteq E(G)$. Let $\mathrm{pm}(G)$ be the number of perfect matchings of G, modulo 2.

For a $V \times V$ matrix A, the *Pfaffian* of A, denoted by $\mathrm{Pf}(A)$, is defined as $\sum_{P\in\mathrm{pair}(V)} \mathrm{sgn}(P) \prod_{xy\in P} a_{x,y}$ where $\mathrm{sgn}(P)$ is the sign of a permutation on the vertices associated with the pairing. As with the determinants, we apply this notion only for adjacency matrices of graphs over $GF(2)$, which means $\mathrm{sgn}(P)$ can be dropped from the formula.

If we evaluate this expression for the adjacency matrix A of a graph G then we obtain the parity of the number of perfect matchings the subgraph in G: the formula determines, for each pairing of $V(G)$, whether or not it is a perfect matching.

For skew matrices (where $a_{u,v} = -a_{v,u}$ for all u, v) it is known that $\text{Pf}(A)^2 = \det(A)$. However, over $GF(2)$ every symmetric matrix is skew, and also the square can be dropped without changing the value. Thus, for a graph G we know that $\det(G) = \text{pm}(G)$.

If we rephrase Theorem 5 we obtain an elegant characterization of the edges after pivoting.

Theorem 10. *Let φ be an applicable sequence of pivoting operations for G, and let $S = \sup(\varphi)$. Then, for vertices x, y with $x \neq y$, $xy \in E(G\varphi)$ iff $\text{pm}(G\langle S \oplus \{x, y\}\rangle) = 1$.*

For small graphs the number of perfect matchings might be easier to determine than the determinant. For instance, for a graph G on four nodes there are only three pairs of edges that can be present to contribute to the value $\text{pm}(G)$.
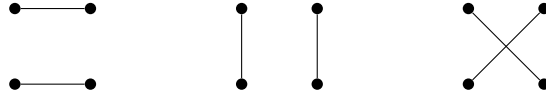
A commutivity result is obtained in [12, Theorem 6.1(iii)]. Assume uv and zw are edges in G on four different vertices u, v, w, z. Then both $[uv][wz]$ and $[wz][uv]$ are applicable iff the induced subgraph $G[\{u, v, w, z\}]$ is not isomorphic to C_4 or D_4.

Its proof in [12] is not difficult, a simple case analysis suffices. Here we note that uv en wz must be edges in order for $[uv]$ and $[wz]$ to be applicable. Both $[uv][wz]$ and $[wz][uv]$ are applicable iff $\text{pm}(G\langle u, v, w, z\rangle) = 1$. Thus the subgraph $G\langle u, v, w, z\rangle$ must contain either one or three perfect matchings, where the first $\{uv, wz\}$ is given. Two perfect matchings occur precisely when the subgraph is isomorphic to C_4 or D_4.

As was noted below Theorem 4, we may also look for a non-empty set S such that every $v \in V(G)$ is adjacent to an even number of vertices of S. E.g., for D_4 we can take S to be the set of the two vertices that are not connected by an edge.

6 Reduced Sequences

We have seen that if we have an applicable sequence of pivots, then the result of that series of operations only depends on the support, the set of vertices occurring an odd number of times as a pivot-vertex. This does not automatically mean that the sequence can be reduced to an equivalent sequence in which each vertex occurs only once. This because one needs to verify that all the operations

are applicable, i.e., that all pivot-pairs are edges in the graph to which they are applied.

We call a sequence of pivots *reduced* [10] if no vertex occurs more than once in the pivots. It turns out that we can use a greedy strategy to reduce a sequence of pivot operations, growing a sequence with given support.

Let G be a graph, and let $S \subseteq V(G)$ be the support of an applicable sequence of pivots for G. We will construct a reduced sequence of pivots with support S. Obviously we may assume that S is non-empty. Observe that since $\det G\langle S\rangle = 1$, there must be at least one element in the adjacency matrix of G that is non-zero, i.e., there is an edge uv in $G\langle S\rangle$.

Apply $[uv]$ to graph G, and proceed iteratively with graph $G[uv]$ and support set $S - \{u, v\}$, where $\det G[uv]\langle D - \{u, v\}\rangle = 1$ again holds (by Theorem 4) and we stop when we have exhausted the support.

Theorem 11. *For every applicable sequence of pivots φ there exists an applicable reduced sequence φ' such that $\sup(\varphi) = \sup(\varphi')$ — and therefore $G\varphi = G\varphi'$.*

Remark 12. The possibility to construct an applicable reduced sequence with given support depends on the fact that there must be at least one edge to obtain a non-zero determinant. In fact every column in the matrix must contain at least one edge. This means we can even choose one of the vertices of the pivot.

As an example, we return to the topic of commutivity. It is known that if $[uv][wz]$ is applicable, then we can not conclude that $[wz][uv]$ is applicable. However, $\det G\langle u, v, w, z\rangle = 1$, so we can construct an applicable sequence with support $\{u, v, w, z\}$. Fixing z we know that there is an edge adjacent to that vertex, which can be either wz, vz or uz. When pivoting over this edge, the remaining two vertices must form an edge in the graph.

Hence, we have shown the following fact: if, for for different vertices, $[uv][wz]$ is applicable, then either at least one of the pivot sequences $[wz][uv]$, $[vz][uw]$, or $[uz][vw]$ is applicable. This is essentially Lemma 1.2.11 of [10]. □

The previous theorem shows that also the converse of Lemma 7 holds.

Theorem 13. *Let S be a set of vertices of graph G. Then $\det G\langle S\rangle = 1$ iff there exists a (reduced) sequence of pivots φ with support S that is applicable in G.*

The size of $\{S \subseteq V \mid \det G\langle S\rangle = 1\}$ is precisely the value of the interlace polynomial $q(G)$ of G on $x = 1$, see [1, Corollary 2].

7   Graphs with Self-Loops

Until now we have considered simple graphs (graphs without loops or parallel edges). In this section we consider graphs G with loops but without parallel edges. The adjacency matrices A corresponding to such graphs are precisely the symmetrical $(0, 1)$-matrices. If vertex u has a loop in G, then the matrix $A\langle\{u\}\rangle$

is equal to the 1×1 matrix (1). Hence, $\det A * \{u\} = 1$ and the general pivot of Section 3 is defined, and is modulo 2 equal to

$$A * \{u\} = \left(\begin{array}{c|c} 1 & \chi_u^T \\ \hline \chi_u & A\langle V - u\rangle - \chi_u \chi_u^T \end{array} \right),$$

where χ_u is the column vector belonging to u without element a_{uu}. We define the elementary pivot $G * u$ for loop vertex u on G by the graph corresponding to adjacency matrix $A * \{u\}$. The elementary pivot $G * u$ is obtained from G by complementing the neighbourhood $N_G(u)$ of u (just as in simple graphs) *and*, for $v \in N_G(u)$, we add a loop to v if v is a non-loop vertex in G and remove the loop if v is a loop vertex in G. Hence, we will call $G * u$ *local complementation* (on graph G with loop u). We can apply Proposition 3, and similar to Theorem 4 we obtain (in $GF(2)$) the following result.

Theorem 14. *Let G be a graph, and let $u \in V(G)$ be a vertex that has a loop. Then, for $Y \subseteq V(G)$,*

$$\det((G * u)\langle Y\rangle) = \det(G\langle Y \oplus \{u\}\rangle)$$

$\square$

The pivot operation on edge e for graph with loops is identical to that operation for simple graphs: it is only defined if both vertices of e do not have loops and it does not remove or add any loop of the graph.

Results of the previous sections carry over to sequences φ of operation having both local complementation and pivot operations. In particular, Theorems 6 and 13 carries over.

Theorem 15. *If φ and φ' are applicable sequences for G having (possibly) both local complementation and pivot operations, then $\sup(\varphi) = \sup(\varphi')$ implies $G\varphi = G\varphi'$. Also, let $S \subseteq V(G)$. Then $\det G\langle S\rangle = 1$ iff there exists a (reduced) sequence φ with support S, having (possibly) both local complementation and pivot operations, that is applicable in G.*

The size of $\{S \subseteq V \mid \det G\langle S\rangle = 1\}$, for graph G with loops, is precisely the value of a polynomial $Q(G)$, defined in [1, Section 4] and related to the interlace polynomial $q(G)$, of G on $x = 2$, see [1, Corollary 5]. Moreover, the previous theorem can also be stated in terms of *general perfect matchings*: considering a loop on x as the edge $\{x\} \in E(G)$, then a general perfect matching is a $P \subseteq E(G)$ that is a partition of $V(G)$.

Remark 16. In the theory of gene assembly in ciliates[6], the local complementation operation on u with the removal of u is called *graph positive rule*, and the pivot operation on uv with the removal of both u and v is called *graph double rule*. These rules are defined on *signed graphs*, where each vertex is labelled by either $-$ or $+$. Now, label $-$ corresponds to a non-loop vertex and label $+$ corresponds to a loop vertex. Hence, we obtain the result that any two sequences

10

of these graph rules with equal support obtain the same graph. Moreover, we obtain that a signed graph can be transformed into the empty graph by these graph rules iff the determinant of corresponding adjacency matrix has determinant 1 modulo 2. □

8 Discussion

We have related applicable sequences of pivot operations to determinants and perfect matchings in a graph. In this way, we have shown that two applicable sequences of pivot operations with equal support have the same effect on the graph. Moreover, for a given set S of vertices, we have shown that there is a applicable sequence φ of pivot operations with support S precisely when the number of perfect matchings of the subgraph induced by S is odd (or equivalently, when the determinant of the adjacency matrix of the subgraph is odd). In fact, there is an applicable reduced sequence φ' with equal support as φ. Finally, we have shown that pivots and local complementation can 'work together' in the case of graphs with loops in the sense that equal support renders equal graphs.

References

1. M. Aigner, H. van der Holst. Interlace polynomials. Linear Algebra and its Applications 377 (2004) 11-30. doi: http://dx.doi.org/10.1016/j.laa.2003.06.010

2. R. Arratia, B. Bollobás, G.B. Sorkin. The interlace polynomial of a graph. Journal of Combinatorial Theory Series B, 92 (2004) 199–233. doi: http://dx.doi.org/10.1016/j.jctb.2004.03.003

3. R. Brijder, T. Harju, I. Petre. Commutativity of the gdr-operation. Erratum to [12]. 2008, unpublished.

4. R.W. Cottle, J.-S. Pang, R.E. Stone. *The Linear Complementarity Problem*. Academic Press, San Diego, 1992.

5. L.E. Danielsen, M.G. Parker. Edge local complementation and equivalence of binary linear codes. Designs, Codes and Cryptography 49 (2008) 161–170. doi: http://dx.doi.org/10.1007/s10623-008-9190-x

6. A. Ehrenfeucht, T. Harju, I. Petre, D.M. Prescott, G. Rozenberg. *Computation in Living Cells – Gene Assembly in Ciliates*. Springer Verlag, 2004.

7. F. Gavril. Algorithms on circular-arc graphs. Networks 4 (1974) 357–369. doi: http://dx.doi.org/10.1002/net.3230040407

8. J.F. Geelen. A generalization of Tutte's characterization of totally unimodular matrices. Journal of Combinatorial Theory, Series B, 70 (1997) 101–117. doi: http://dx.doi.org/10.1006/jctb.1997.1751

9. J.F. Geelen. Matchings, matroids and unimodular matrices. Ph.D. Thesis (1995), available online on author's website.

10. F. Genest. Graphes eulériens et complémentarité locale. Ph.D. Thesis, Université de Montréal, 2002. arXiv:math/0701421v1 [math.CO]

11. S. Hannenhalli, P.A. Pevzner. Transforming cabbage into turnip: polynomial algorithm for sorting signed permutations by reversals. Journal of the ACM 46 (1999) 1–27. doi: http://dx.doi.org/10.1145/300515.300516

12. T. Harju, C. Li, I. Petre, G. Rozenberg. Parallelism in Gene Assembly. Natural Computing 5 (2006) 203–223. doi: http://dx.doi.org/10.1007/s11047-005-4462-0

13. C.H.C. Little. The parity of the number of 1-factors of a graph, Discrete Mathematics 2 (1972) 179–181. doi: http://dx.doi.org/10.1016/0012-365X(72)90083-0

14. M. van den Nest, B. de Moor. Edge-local equivalence of graphs. arXiv:math/0510246

15. S. Oum. Rank-width and vertex-minors. Journal of Combinatorial Theory, Series B, 95 (2005) 79–100. doi: http://dx.doi.org/10.1016/j.jctb.2005.03.003

16. A.W. Tucker. A combinatorial equivalence of matrices. Combinatorial Analysis, Proceedings of symposia in applied mathematics, volume 10, 129-140. AMS, 1960.

A Pivots and Matchings

In this appendix we give an independent proof of Theorem 10 in the style of Oum [15], using perfect matchings instead of determinants, as it may be of independent interest. The proof was made superfluous when the authors discovered references [13] and [8]. However, the proofs in this appendix are straightforward and therefore the reader may prefer this approach.

Recall that $x \sim_G y$ if either $xy \in E(G)$ or $x = y$.

As a technical tool we need a formula that can be used to compute the number of perfect matchings in a graph, but which can also be applied when we have duplicate vertices.

For (an even number of) variables $x_1, \ldots, x_n$ let $\mathrm{pm}_G(x_1, \ldots, x_n)$ denote the following logical expression:

$$\bigoplus_{P \in \mathrm{pair}\{x_1, \ldots, x_n\}} \bigwedge_{xy \in P} (x \sim_G y)$$

The number of variables used in the expression varies; we assume this number is clear from the context. Clearly, $\mathrm{pm}_G(x, y)$ equals $x \sim_G y$. Moreover $\mathrm{pm}_G()$ is true – the logical and $\wedge$ over 0 arguments is (considered) true, and the logical exclusive or over 1 argument a is (considered) a. This is in line with the fact that there is a single perfect matching on zero vertices.

If we evaluate this expression for the (pairwise different) vertices $v_1, \ldots, v_n$ of graph G then we obtain the value $\mathrm{pm}_G(v_1, \ldots, v_n)$ which equals $\mathrm{pm}(G\langle\{v_1, \ldots, v_n\}\rangle)$, the parity of the number of perfect matchings the subgraph in G induced by $v_1, \ldots, v_n$ (identifying 0 and 1 with false and true, respectively). Due to the highly symmetric form of the formula pm_G the ordering of the vertices as arguments to the formula is not important for the value. We will use this fact frequently below.

The formula can also be evaluated when two (or more) of its arguments are chosen to be the same vertex in the graph. The next result shows equal vertices can be omitted (in pairs).

Lemma 17. *Let $v_1, \ldots, v_{n-2}, v, v'$ be vertices in graph G such that $v \sim_G x$ iff $v' \sim_G x$ for each vertex x. Then $\mathrm{pm}_G(v_1, \ldots, v_{n-2}, v, v') = \mathrm{pm}_G(v_1, \ldots, v_{n-2})$.*

Proof. Observe that the condition of the lemma on v and v' implies that $v \sim_G v'$ holds.

For $n = 2$ the left hand side $\mathrm{pm}_G(v, v')$ equals $v \sim_G v'$ which equals the right hand side $\mathrm{pm}_G()$ which has been set to true.

Now let $n > 2$. In the formal expression pm_G each pairing P that does not contain $x_{n-1}x_n$ has two pairs $x_{n-1}x_i$ and x_nx_j. For P there is a (unique) P' corresponding to P where $P' \backslash P = \{x_{n-1}x_j, x_nx_i\}$ (and hence $P \backslash P' = \{x_{n-1}x_i, x_nx_j\}$). Since $v \sim_G x$ iff $v' \sim_G x$ for each vertex x, we have $(v \sim_G x_i) \wedge (v' \sim_G x_j) = (v \sim_G x_j) \wedge (v' \sim_G x_i)$. Hence the contributions of pairings P and P' cancel.

The remaining pairings all contain vv' for which $v \sim_G v'$ can be dropped from the formula, as $v \sim_G v'$ holds. The resulting formula equals that of $\mathrm{pm}_G(v_1, \ldots, v_{n-2})$.

$\square$

The next lemma shows that we can characterize pivoting by the parity of the number of perfect matchings in subgraphs. It is a simple reformulation of the result of Oum, but essential as a first step to understand the connection between pivoting and perfect matchings.

Lemma 18. *Let $G = (V, E)$ be a graph, and fix $uv \in E$. For $x, y \in V$ we have $\mathrm{pm}_{G[uv]}(x, y) = \mathrm{pm}_G(x, y, u, v)$.*

Proof. In the evaluation of $\mathrm{pm}_{G[uv]}(x, y)$ we consider a single pair $x \sim_{G[uv]} y$ only, the left-hand side in Lemma 2.

As $u \sim_G v$ holds, we may replace the factor $x \sim_G y$ in the statement of Lemma 2 by $(x \sim_G y \wedge u \sim_G v)$. Now the right-hand side of the formula equals $\mathrm{pm}_G(x, y, u, v)$.

$\square$

The main technical result is a generalization of the previous lemma, which now includes an additional sequence of nodes on both sides. Before stating this result we explicitly compute the simplest of these generalizations, with variables x_1, x_2, x_3, x_4 instead of x, y. This example visualizes the more general argumentations in the proof of our general result, which follows the example.

Example 19. $\mathrm{pm}_{G[uv]}(x_1, x_2, x_3, x_4)$ equals $(x_1 \sim_{G[uv]} x_2 \wedge x_3 \sim_{G[uv]} x_4) \oplus (x_1 \sim_{G[uv]} x_3 \wedge x_2 \sim_{G[uv]} x_4) \oplus (x_1 \sim_{G[uv]} x_4 \wedge x_2 \sim_{G[uv]} x_3)$.

Now substitute each $x \sim_{G[uv]} y$ by the formula given in Lemma 2, to obtain
$(\, [x_1x_2 \oplus (x_1u \wedge x_2v) \oplus (x_1v \wedge x_2u)] \wedge [x_3x_4 \oplus (x_3u \wedge x_4v) \oplus (x_3v \wedge x_4u)] \,) \bigoplus (\, [x_1x_3 \oplus (x_1u \wedge x_3v) \oplus (x_1v \wedge x_3u)] \wedge [x_2x_4 \oplus (x_2u \wedge x_4v) \oplus (x_2v \wedge x_4u)] \,) \bigoplus (\, [x_1x_4 \oplus (x_1u \wedge x_4v) \oplus (x_1v \wedge x_4u)] \wedge [x_2x_3 \oplus (x_2u \wedge x_3v) \oplus (x_2v \wedge x_3u)] \,)$,
where we write xy rather than $x \sim_G y$.

By distributivity (i.e. using the logical identity $a \wedge (b \oplus c) = (a \wedge b) \oplus (a \wedge c)$) this is equivalent to
$(x_1x_2 \wedge x_3x_4) \oplus (x_1x_2 \wedge x_3u \wedge x_4v) \oplus (x_1x_2 \wedge x_3v \wedge x_4u) \oplus (x_1u \wedge x_2v \wedge x_3x_4) \oplus (x_1u \wedge x_2v \wedge x_3u \wedge x_4v) \oplus (x_1u \wedge x_2v \wedge x_3v \wedge x_4u) \oplus (x_1v \wedge x_2u \wedge x_3x_4) \oplus (x_1v \wedge x_2u \wedge x_3u \wedge x_4v) \oplus (x_1v \wedge x_2u \wedge x_3v \wedge x_4u) \bigoplus (x_1x_3 \wedge x_2x_4) \oplus (x_1x_3 \wedge x_2u \wedge x_4v) \oplus (x_1x_3 \wedge x_2v \wedge x_4u) \oplus (x_1u \wedge x_3v \wedge x_2x_4) \oplus (x_1u \wedge x_3v \wedge x_2u \wedge x_4v) \oplus (x_1u \wedge x_3v \wedge x_2v \wedge x_4u) \oplus (x_1v \wedge x_3u \wedge x_2x_4) \oplus (x_1v \wedge x_3u \wedge x_2u \wedge x_4v) \oplus (x_1v \wedge x_3u \wedge x_2v \wedge x_4u) \bigoplus (x_1x_4 \wedge x_2x_3) \oplus (x_1x_4 \wedge x_2u \wedge x_3v) \oplus (x_1x_4 \wedge x_2v \wedge x_3u) \oplus (x_1u \wedge x_4v \wedge x_2x_3) \oplus (x_1u \wedge x_4v \wedge x_2u \wedge x_3v) \oplus (x_1u \wedge x_4v \wedge x_2v \wedge x_3u) \oplus (x_1v \wedge x_4u \wedge x_2x_3) \oplus (x_1v \wedge x_4u \wedge x_2u \wedge x_3v) \oplus (x_1v \wedge x_4u \wedge x_2v \wedge x_3u)$

There are twelve terms with four factors, which are six different terms each occurring twice, hence cancelling each other. The three terms with two factors can be extended adding a third term uv (which is true). Rearranging these 15 remaining terms we get
$(x_1x_2 \wedge x_3x_4 \wedge uv) \oplus (x_1x_2 \wedge x_3u \wedge x_4v) \oplus (x_1x_2 \wedge x_3v \wedge x_4u) \oplus (x_1x_3 \wedge x_2x_4 \wedge uv) \oplus (x_1x_3 \wedge x_2u \wedge x_4v) \oplus (x_1x_3 \wedge x_2v \wedge x_4u) \oplus (x_1x_4 \wedge x_2x_3 \wedge uv) \oplus (x_1x_4 \wedge x_2u \wedge$

14

$x_3 v) \oplus (x_1 x_4 \wedge x_2 v \wedge x_3 u) \oplus (x_1 u \wedge x_2 v \wedge x_3 x_4) \oplus (x_1 u \wedge x_3 v \wedge x_2 x_4) \oplus (x_1 u \wedge x_4 v \wedge x_2 x_3) \oplus (x_1 v \wedge x_2 u \wedge x_3 x_4) \oplus (x_1 v \wedge x_3 u \wedge x_2 x_4) \oplus (x_1 v \wedge x_4 u \wedge x_2 x_3)$

These happen to be the fifteen pairings making up $\mathrm{pm}_G(x_1, x_2, x_3, x_4, u, v)$.

$\square$

As announced, the proof of our general result follows the path sketched in the previous example. It is the 'perfect matching counterpart' of Theorem 4.

Theorem 20. *Let G be a graph, let $v_1, \ldots, v_n$ be vertices in G, let $uv \in E(G)$. Then $\mathrm{pm}_{G[uv]}(v_1, \ldots, v_n) = \mathrm{pm}_G(v_1, \ldots, v_n, u, v)$.*

Proof. If $n = 0$ the left hand side equals $\mathrm{pm}_{G[uv]}()$ which is true, while the right hand side $\mathrm{pm}_G(u, v)$ is equivalent to $u \sim_G v$, which is also true, as uv is an edge in G.

Now let $n \geq 2$. For $\mathrm{pm}_{G[uv]}(v_1, \ldots, v_n)$ the following formula has to be evaluated

$$\bigoplus_{P \in \mathrm{pair}\{x_1, \ldots, x_n\}} \bigwedge_{xy \in P} (x \sim_{G[uv]} y)$$

According to Lemma 2 the relation $\sim_{G[uv]}$ can be replaced by a suitable expression involving $\sim_G$ in the original graph G.

$$\bigoplus_{P \in \mathrm{pair}\{x_1, \ldots, x_n\}} \bigwedge_{xy \in P} ((x \sim_G y) \oplus (x \sim_G u \wedge y \sim_G v) \oplus (x \sim_G v \wedge y \sim_G u))$$

Now, we apply the logical identity $a \wedge (b \oplus c) = (a \wedge b) \oplus (a \wedge c)$ iteratively to the inner part $\bigwedge_{xy \in P}(\ldots)$, and we obtain for each $P \in \mathrm{pair}\{x_1, \ldots, x_n\}$ the exclusive or over a total of $3^{n/2}$ terms, each of which is a conjunction of factors of one of the forms $x \sim_G y$, $(x \sim_G u \wedge y \sim_G v)$ and $(x \sim_G v \wedge y \sim_G u)$. Moreover in each such term the variables $x_1, \ldots, x_n$ each occur exactly once.

Now consider such a term in which the constant u occurs k times paired to $x_{i_1}, \ldots, x_{i_k}$, which implies also v occurs k times paired to certain $x_{j_1}, \ldots, x_{j_k}$.

Up to the order of factors, this term is present in the list that belongs to any P' that pairs the variables $x_{i_1}, \ldots, x_{i_k}$, to the $x_{j_1}, \ldots, x_{j_k}$ (in any combination) and equals P for the other variables. There are $k!$ such pairings, thus $k!$ copies of equivalent terms. These copies cancel if $k!$ is even, which means if $k \geq 2$.

Hence, for each P we need only consider those terms for which there is at most one occurrence of both u and v. Thus we have reduced the previous equation to

$$\bigoplus_{P \in \mathrm{pair}\{x_1, \ldots, x_n\}} \left(\bigwedge_{x_1 y_1 \in P} x_1 \sim_G y_1 \right) \oplus$$
$$\bigoplus_{x_1 y_1 \in P} \left[\left(x_1 \sim_G u \wedge y_1 \sim_G v \wedge \bigwedge_{xy \in P \setminus x_1 y_1} x \sim_G y \right) \right.$$
$$\left. \oplus \left(x_2 \sim_G u \wedge x_1 \sim_G v \wedge \bigwedge_{xy \in P \setminus x_1 x_2} x \sim_G y \right) \right]$$

Because $\bigwedge_{xy \in P} x \sim_G y = \bigwedge_{xy \in P \cup \{uv\}} (x \sim_G y)$, this is equivalent to

$$\bigoplus_{P \in \mathrm{pair}\{x_1, \ldots, x_n, u, v\}} \left(\bigwedge_{xy \in P} x \sim_G y \right)$$

and this in turn is the expression that has to be evaluated for $\mathrm{pm}_G(x_1, \ldots, x_n, u, v)$.

□

By Lemma 17, the previous theorem may be rephrased as follows, cf. Theorem 4.

Theorem 21. *Let G be a graph, and let $uv \in E(G)$. Then, for $Y \subseteq V(G)$,*

$$\mathrm{pm}((G[uv])\langle Y \rangle) = \mathrm{pm}(G\langle Y \oplus \{u, v\}\rangle)$$

□

The results of Section 4 involving $\det(G)$ can hence also be developed using $\mathrm{pm}(G)$ through Theorem 21. Hence, we obtain, e.g., Theorem 10.

The following special case of our general result Theorem 20 is a reformulation in the style of the original Lemma 2, summing over edges in the subgraph of G induced by $\{u, v, w, z\}$ (with some care in the case of multiple occurrences of vertices).

Theorem 22. *If $[uv][wz]$ is applicable to G, then*

$$x \sim_{G[uv][wz]} y = x \sim_G y \bigoplus_{\substack{\{x_1 x_2, x_3 x_4\} \\ \textit{pairing of} \\ \{u, v, w, z\} \\ \textit{with } x_1 \sim_G x_2}} ((x \sim_G x_3) \wedge (y \sim_G x_4)) \oplus ((x \sim_G x_4) \wedge (y \sim_G x_3))$$

Proof. The result is obtained by rewriting the expression for $\mathrm{pm}_G(x, y, u, v, w, z)$, and using the fact that $\mathrm{pm}_G(u, v, w, z)$ holds. □